EXTREMAL PROPERTIES OF THE SOLUTIONS OF CERTAIN CLASSES OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS, AND THEIR APPLICATION TO THE FLOW OF A GAS

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In the investigation and solution of flows of compressible and incompressible fluids, there may be great value in a preliminary examination of the extremal properties of the solutions of the differential equations of the given problem, with or without the boundary conditions. In this way, many important facts can be obtained, for instance, theorems concerning the attainment of extremal values for perturbations of steady potential flows directly on the boundaries of the bodies. On the other hand, knowing the extremal properties of the solution, it is possible, in a number of cases, to develop simple, rapidly converging, computational methods.

In many cases, the investigation of the extremal properties of the solution can be accomplished by proceeding directly from the properties of the differential equations.

In the following, we derive a theorem concerning the extremal properties of a given class of equations of second order, and demonstrate its application to gas dynamics.

Theorem. If the differential operator

$$L(u) = \Phi(x, y, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$$

is such that its application to any twice continuously-differentiable function $f(x, y)$ at its point of maximum, N_0, and in some neighborhood of that point gives a non-positive value, and $\Phi_{u_{xx}}(N_0) > 0$, or $\Phi_{u_{yy}}(N_0) > 0$, then the solution $u(x, y)$ of the equation $L(u) = 0$ cannot have a maximum value at that point or in a certain neighborhood of it.

We require that the function Φ be continuous and continuously differentiable with respect to u, u_x and u_{xx} or to u, u_y and u_{yy} in some region of variation of its arguments.

In that case, if application of the operator to a twice continuously differentiable function at its point of minimum gives a non-negative value, and the preceding requirements are satisfied, then the conclusion of the theorem is that the solution of the equation $L(u) = 0$ cannot attain a minimum.

Proof. Assume that the solution $u(x, y)$ of the equation $L(u) = 0$ attains a maximum at some point $N_0(x_0, y_0)$, at which the conditions of the theorem are satisfied. Choose a neighborhood G of the point N_0 such that inside it the stated requirements are also fulfilled, and introduce the function

$$u^* (x, y) = u (x, y) + k (x - x_0)^2 \qquad (k > 0)$$

which, for sufficiently small k, will also reach a maximum at some point $N_1(x_1, y_1)$ of the chosen neighborhood. It is sufficient to take

$$k < \frac{|M| - |m|}{\delta^2}$$

where $M = u(N_0)$, $m = \sup u(N)$ on the boundary Γ of the region G around the point N_0, and δ is the diameter of region G.

Applying the operator L to the function u^* at the point N_1 should, according to the conditions, give a non-negative value

$$L [u^* (N_1)] \leqslant 0$$

On the other hand

$$L [u^* (N_1)] = \Phi (x_1, y_1, u_1 + k (x_1 - x_0)^2, u_{x_1} + 2k (x_1 - x_0), u_{y_1}, u_{xy_1}, u_{yy_1}, u_{xx} + 2k)$$

In view of the conditions of the theorem, we can apply Lagrange's theorem of the mean to the right-hand side (taking $\Phi_{u_{xx}}(N_0) > 0$ and Φ to be continuously differentiable with respect to u, u_x and u_{xx}).

The result is

$$L [u^* (N_1)] = \Phi (x_1, y_1, u_1, u_{x_1}, u_{y_1}, u_{xy_1}, u_{yy_1}, u_{xx_1}) +$$
$$+ \Phi_u (x_1, y_1, u_1 + \theta k (x_1 - x_0)^2, \quad u_{x_1} + 2\theta k (x_1 - x_0), u_{y_1}, u_{xy_1}, u_{yy_1}, u_{xx_1} + 2\theta k) \times$$

$$\times k (x_1 - x_0)^2 +$$
$$+ \Phi_{u_x}(x_1, y_1, u_1 + \theta k (x_1 - x_0)^2, u_{x_1} + 2\theta k (x_1 - x_0), u_{y_1}, u_{xy_1}, u_{yy_1}, u_{xx_1} + 2\theta k) \times$$
$$\times 2K (x_1 - x_0) +$$
$$+ \Phi_{u_{xx}}(x_1, y_1, u_1 + \theta k (x_1 - x_0)^2, u_{x_1} + 2\theta k (x_1 - x_0), u_{y_1}, u_{xy_1}, u_{yy_1}, u_{xx_1} + 2\theta k)\, 2k$$
$$(0 < \theta < 1)$$

The first term of the right-hand side is equal to zero, since $u(x,y)$ is the solution of the equation $L(u) = \Phi(x, y, u, u_x, u_y, u_{xy}, u_{yy}, u_{xx}) = 0$. The next two terms are smaller to a higher order than the last one. Shrinking the region G to the point N_0, $x_1 \to x_0$, $y_1 \to y_0$. In view of the theorem, $\Phi_{u_{xx}} > 0$ at the point N_0 and the whole expression becomes positive. The resulting contradiction proves the theorem.

Instead of the function $u^*(x, y) = u(x, y) + k(x - x_0)^2$, we may take the function $u^{**}(s, y) = u(x, y) + k(y - y_0)^2$, which may solve the problem if it is difficult or impossible to determine the sign of $\Phi_{u_{xx}}(N_0)$. Here, it will be necessary to require that Φ be continuously differentiable with respect to u, u_y and u_{yy}.

The second part of the theorem, concerning the minimum, can be proved by introducing the function

$$u^{***} = u(x, y) - k(x - x_0)^2$$

Let us apply the results obtained to equations of motion of a gas (cf., for example, [1]). The requirements of the theorem are satisfied, for example, by the equations for the velocity potential, φ, for plane, axisymmetric, and three-dimensional flows of a gas, which have, respectively, the forms

$$P(\varphi) = [(k+1)(a_*^2 - \varphi_x^2) - (k-1)\varphi_y^2]\varphi_{xx} + \\ + [(k+1)(a_*^2 - \varphi_y^2) - (k-1)\varphi_x^2]\varphi_{yy} - 4\varphi_x\varphi_y\varphi_{xy} = 0 \tag{1}$$
$(x, y$ — Cartesian coordinates$)$

$$Q(\varphi) = [(k+1)(a_*^2 - \varphi_x^2) - (k-1)\varphi_r^2]\varphi_{xx} + \\ + [(k+1)(a_*^2 - \varphi_r^2) - (k-1)\varphi_x^2]\varphi_{rr} - 4\varphi_x\varphi_r\varphi_{xr} + \\ + [(k+1)a_*^2 - (k-1)(\varphi_x^2 + \varphi_r^2)]\frac{\varphi_r}{r} = 0 \tag{2}$$
$(x, z$ — cylindrical coordinates$)$

$$R(\varphi) = [(k+1)(a_*^2 - \varphi_x^2) - (k-1)(\varphi_y^2 + \varphi_z^2)]\varphi_{xx} + \\ + [(k+1)(a_*^2 - \varphi_y^2) - (k-1)(\varphi_z^2 + \varphi_x^2)]\varphi_{yy} + \\ + [(k+1)(a_*^2 - \varphi_z^2) - (k-1)(\varphi_x^2 + \varphi_y^2)]\varphi_{zz} - \\ - 4\varphi_x\varphi_y\varphi_{xy} - 4\varphi_y\varphi_z\varphi_{yz} - 4\varphi_z\varphi_x\varphi_{zx} = 0 \tag{3}$$
$(x, y, z$ — Cartesian coordinates$)$

Here, a^* is the critical speed and k the adiabatic index.

For the cases enumerated, the role of a twice continuously differentiable function $f(x, y)$ is played by the solution of the equation itself, namely the velocity potential. It follows that their solutions do not attain maxima and minima in the flow field.

For plane flows of a gas, this result was obtained in a different way by Chaplygin [2]. This conclusion turns out to be valid also for the stream functions of plane and axisymmetric flows. For example, the stream function ψ for an axisymmetric section satisfies the equation

$$R(\psi) = \left[\left(\frac{\rho a}{\rho_0}\right)^2 - \frac{\psi_r^2}{r^2}\right]\psi_{xx} + \left[\left(\frac{\rho a}{\rho_0}\right)^2 - \frac{\psi_x}{r^2}\right]\psi_{rr} + \frac{2}{r^2}\psi_x\psi_r\psi_{xr} - \left(\frac{\rho a}{\rho_0}\right)^2\frac{\psi_r}{r} = 0$$

Here, ρ and ρ_0 are densities, a is the speed of sound and w the magnitude of the velocity

$$\frac{\rho}{\rho_0} = \left(1 - \frac{k-1}{k+1}\frac{w^2}{a_*^2}\right)^\beta \qquad \left(\beta = \frac{1}{k-1}\right)$$

$$a^2 = \frac{1}{2}[(k+1)a_*^2 - (k-1)w^2$$

$$w^2 = \varphi_x^2 + \varphi_r^2, \qquad w^2\left(1 - \frac{k-1}{k+1}\frac{w^2}{a_*^2}\right)^{2\beta} = \frac{\psi_x^2 + \psi_r^2}{r^2}$$

and the connection between the stream function and the velocity potential is given by the relations

$$\psi_r = \left(1 - \frac{\varphi_x^2 + \varphi_r^2}{\beta(k+1)a_*^2}\right)^\beta \varphi_x, \qquad \psi_x = -\left(1 - \frac{\psi_x^2 + \varphi_r^2}{\beta(k+1)a_*^2}\right)^\beta \varphi_r$$

Using these relations, the equation for the stream function can be put in the form

$$R(\psi) = [b + f_1(\psi_x, \psi_r)]\psi_{xx} + [b + f_2(\psi_x, \psi_r)]\psi_{rr} +$$
$$+ \frac{2}{r^2}\psi_x\psi_r\psi_{xr} + [b + f_3(\psi_x, \psi_r)]\frac{\psi_r}{r} = 0 \qquad (b > 0)$$

Here, f_1, f_2 and f_3 are known functions, which become zero for $\psi_x = \psi_y = 0$.

The applicability of the given theorem to the stream function follows immediately from this relation.

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